International Journal of Theoretical Physics, Vol. 7, No. 4 (1973), pp. 287–289

Adjoint Solutions of the Brans-Dicke Equations

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Received: 8 *May* 1972

Abstract

It is shown that if the Brans–Dicke equations have the solution ϕ, g_{ij} generated by the trace-free source T_{kl} $(T = 0)$ then there exists an 'adjoint solution' ϕ^{-1}, $\phi^2 g_{ij}$ of these equations generated by the source $\phi^{-2} T_{kl}$. An example is considered.

1. *Introduction*

The field equations of the scalar-tensor theory of gravitation due to Brans & Dicke (1961) are

$$G_{ij} = 8\pi\phi^{-1} T_{ij} + \omega\phi^{-2}(\phi_{;i}\,\phi_{;j} - \tfrac{1}{2}g_{ij}\,\phi_{;k}\,\phi^{;k}) + \phi^{-1}(\phi_{;ij} - g_{ij}\,\Box\,\phi) \tag{1.1}$$

$$2\phi^{-1}\,\Box\,\phi - \phi^{-2}\,\phi_{;k}\,\phi^{;k} = \omega^{-1} R \tag{1.2}$$

the speed of light having been taken to have the value unity. (1.2) may be replaced by the equation

$$(3 + 2\omega)\,\Box\,\phi = 8\pi T \tag{1.3}$$

Now, although conformal transformations of the metric tensor g_{ij} have occasionally been considered before, for instance by Dicke (1962) and Bergmann (1968), the following result has not, as far as I am aware, been stated previously:

> if $(T_{kl}|\phi, g_{ij})$ is a solution of the Brans–Dicke equations with trace-free source T_{kl} $(T = 0)$ then the equations are also satisfied by $(\phi^{-2} T_{kl}|\phi^{-1}, \phi^2 g_{ij})$.

In particular, the adjoint solution may evidently be formed when one has an electromagnetic source.

The derivation of the general result is given in Section 2, whilst in Section 3 the static spherically symmetric solution of Brans (1962) is examined in the present context.

2. *Proof of the Result*

Since T vanishes by hypothesis,

$$\Box\phi = 0 \tag{2.1}$$

granted that $2\omega + 3 \neq 0$. Then (1.1) may be replaced by the simpler equation

$$8\pi T_{ij} = -\phi R_{ij} - \omega\phi^{-1}\phi_{;i}\phi_{;j} - \phi_{;ij} \tag{2.2}$$

Now let

$$\bar{g}_{ij} = \phi^2 g_{ij}, \qquad \bar{\phi} = \phi^{-1} \tag{2.3}$$

Further, any quantity relating to the Riemann space $\bar{V}_4$ whose metric tensor is $\bar{g}_{ij}$ is distinguished by a bar, and covariant derivatives in this $\bar{V}_4$ are denoted by indices following a colon. Then, if s is any scalar field

$$\begin{aligned} s_{:ij} &= s_{,ij} - \bar{\Gamma}^k{}_{ij} s_{,k} \\ &= s_{;ij} - \phi^{-1}(s_{;i}\phi_{;j} + s_{;j}\phi_{;i} - g_{ij}s^{;k}\phi_{;k}) \end{aligned} \tag{2.4}$$

In particular, taking $s = \bar{\phi}$,

$$\bar{\phi}_{:ij} = -\phi^{-2}\phi_{;ij} + \phi^{-3}(4\phi_{;i}\phi_{;j} - g_{ij}\phi_{;k}\phi^{;k}) \tag{2.5}$$

Hence

$$\bar{\Box}\bar{\phi} = \phi^{-2}g^{ij}\bar{\phi}_{:ij} = -\phi^{-4}\Box\phi = 0 \tag{2.6}$$

in view of (2.1). With this result at hand, the source belonging to the fields $\bar{\phi}$, $\bar{g}_{ij}$ is given by

$$8\pi\bar{T}_{ij} = -\bar{\phi}\bar{R}_{ij} - \omega\bar{\phi}^{-1}\bar{\phi}_{:i}\bar{\phi}_{:j} - \bar{\phi}_{:ij} \tag{2.7}$$

The Ricci tensor of the $\bar{V}_4$ is (Eisenhart, 1949)

$$\bar{R}_{ij} = R_{ij} + 2\phi^{-1}\phi_{;ij} - 4\phi^{-2}\phi_{;i}\phi_{;j} + g_{ij}\phi^{-2}\phi_{;k}\phi^{;k} \tag{2.8}$$

once again bearing (2.1) in mind. If one now inserts (2.5) and (2.8) in (2.7) one just recovers T_{ij} (as given by (2.2)) to within a factor ϕ^{-2}:

$$\bar{T}_{ij} = \phi^{-2}T_{ij} \tag{2.9}$$

and the desired result has thus been attained.

3. *The Static Spherically Symmetric Vacuum Solution as Example*

Brans (1962) has given the exact static spherically symmetric vacuum solution ($T_{ij} = 0$) of equations (1.1, 1.2) in isotropic coördinates:

$$ds^2 = -e^{\lambda}[dr^2 + r^2(d\theta_1{}^2 + \sin^2\theta_1\, d\phi_1{}^2)] + e^{\nu}\, dt^2$$

with

$$\begin{aligned} e^{\lambda} &= e^{\lambda_0}(1 + B/r)^{2+2(C+1)/A}(1 - B/r)^{2-2(C+1)/A} \\ e^{\nu} &= e^{\nu_0}(1 + B/r)^{-2/A}(1 - B/r)^{2/A} \\ \phi &= \phi_0(1 + B/r)^{-C/A}(1 - B/r)^{C/A} \end{aligned} \tag{3.1}$$

B, C, λ_0, ν_0, ϕ_0 are constants of integration, and

$$A^2 = (1 + \tfrac{1}{2}\omega)C^2 + C + 1 \tag{3.2}$$

(It suffices to consider only the first of the four 'branches' given by Brans.) Although the situation here is a very special one, both on account of the many symmetries present and because T_{ij} actually vanishes, it is still worth contemplating it in the context of the present investigation. Of course, since (3.1) is a 'general' solution the adjoint solution must be already contained in it. At any rate,

$$\begin{aligned} e^{\bar\lambda} &= e^{\lambda_0}(1 + B/r)^{2+2/A}(1 - B/r)^{2-2/A} \\ e^{\bar\nu} &= e^{\nu_0}(1 + B/r)^{-2(C+1)/A}(1 - B/r)^{2(C+1)/A} \\ \bar\phi &= \phi_0^{-1}(1 + B/r)^{C/A}(1 - B/r)^{-C/A} \end{aligned} \tag{3.3}$$

If, however, one writes $e^{\bar\lambda}$, $e^{\bar\nu}$, $\bar\phi$ in the form (3.1) with all constants of integration barred, one has expressions just of the form (3.3), i.e. with

$$\bar\lambda_0 = \lambda_0, \qquad \bar\nu_0 = \nu_0, \qquad \bar\phi_0 = \phi_0^{-1}, \qquad \bar B = B, \qquad \bar A = A/(C+1), \\ \bar C = -C/(C+1) \tag{3.4}$$

and this is in harmony with the remark preceding equation (3.3).

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